A Measurable-Group-Theoretic Solution to von Neumann's Problem

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March 11, 2009

Abstract

We give a positive answer, in the measurable-group-theory context, to von Neumann's problem of knowing whether a non-amenable countable discrete group contains a non-cyclic free subgroup. We also get an embedding result of the free-group von Neumann factor into restricted wreath product factors.

2000 Mathematical Subject Classification: 37A20, 20E05, 82B43, 05C80

Key words and phrases: Orbit equivalence, free group, non-amenability, von Neumann's problem, percolation theory, graphs, trees

Amenability of groups is a concept introduced by J. von Neumann in his seminal article [vN29] to explain the so-called Banach-Tarski paradox. He proved that a discrete group containing the free group $\mathbf{F}_2$ on two generators as a subgroup is non-amenable. Knowing whether this was a characterization of non-amenability became known as von Neumann's Problem and was solved by the negative by A. Ol′šanskiĭ [Ol′80]. Still, this characterization could become true after relaxing the notion of "containing a subgroup". K. Whyte gave a very satisfying geometric group-theoretic solution: *A finitely generated group Γ is non-amenable iff it admits a partition with pieces uniformly bilipschitz equivalent to the regular 4-valent tree [Why99].* Geometric group theory admits a measurable counterpart, namely, measurable group theory. The main goal of our note is to provide a solution to von Neumann's problem in this context. We show that any countable non-amenable group admits a measure-preserving free action on some probability space, such that the orbits may be measurably partitioned into pieces given by an $\mathbf{F}_2$-action.

To be more precise, we use the following notation. For a finite or countable set M, let μ denote the product $\otimes_M Leb$ on $[0,1]^M$ of the Lebesgue measures Leb on $[0,1]$, and for $p \in [0,1]$, let μ_p denote the product of the discrete measures $(1-p)\delta_{\{0\}} + p\delta_{\{1\}}$ on $\{0,1\}^M$. Thus, the meaning of μ may vary from use to use as M varies. Usually M will be a countable group Λ or the set $\mathtt{E}$ of edges of a Cayley graph.

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†Supported partially by NSF grant DMS-0705518 and Microsoft Research.

Theorem 1 *For any countable discrete non-amenable group Λ, there is a measurable ergodic essentially free action of $\mathbf{F}_2$ on $([0,1]^\Lambda, \mu)$ such that almost every Λ-orbit of the Bernoulli shift decomposes into $\mathbf{F}_2$-orbits.*

In other words, the orbit equivalence relation of the $\mathbf{F}_2$-action is contained in that of the Λ-action. We give two proofs of this theorem, each with its own advantages.

For some purposes, it is useful to get a Bernoulli shift action with a discrete base space. We show:

Theorem 2 *For any finitely generated non-amenable group Γ, there is $n \in \mathbb{N}$ and a non-empty interval (p_1, p_2) of parameters p for which there is an ergodic essentially free action of $\mathbf{F}_2$ on $\prod_1^n(\{0,1\}^\Gamma, \mu_p)$ such that almost every Γ-orbit of the diagonal Bernoulli shift decomposes into $\mathbf{F}_2$-orbits.*

These results have operator-algebra counterparts:

Corollary 3 *Let Λ be a countable discrete non-amenable group and H be an infinite group. Then the von Neumann factor $L(H \wr \Lambda)$ of the restricted wreath product contains a copy of the von Neumann factor $L(\mathbf{F}_2)$ of the free group.*

Corollary 4 *Let Γ be a finitely generated discrete non-amenable group. Let n, p_1, p_2 be as in Theorem 2 and let $p = \frac{\alpha}{\beta} \in (p_1, p_2)$, with $\alpha, \beta \in \mathbb{N}$. Assume that H contains an abelian subgroup K of order $k = \beta^n$. Then the von Neumann factor $L(H \wr \Lambda)$ of the restricted wreath product contains a copy of the von Neumann factor $L(\mathbf{F}_2)$ of the free group.*

For this paper, we assume a certain familiarity with the results and notation of [Gab05], [Gab00] and [LS99].

Acknowledgment

We are very grateful to Sorin Popa for bringing to our attention the above corollaries. We also thank the referee for a careful reading.

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A ***(countable standard) equivalence relation*** on the standard Borel space (X, ν) is an equivalence relation $\mathcal{R}$ with countable classes that is a Borel subset of $X \times X$ for the product σ-algebra.

A ***(measure-preserving oriented) graphing*** on (X, ν) is a denumerable family $\Phi = (\varphi_i)_{i \in I}$ of partial measure-preserving isomorphisms $\varphi_i : A_i \to B_i$ between Borel subsets $A_i, B_i \subset X$.

A graphing Φ ***generates*** an equivalence relation $\mathcal{R}_\Phi$: the smallest equivalence relation that contains all pairs $(x, \varphi_i(x))$. The ***cost*** of a graphing $\Phi = (\varphi_i)_{i \in I}$ is the sum of the measures of the domains $\sum_{i \in I} \nu(A_i)$. The ***cost***, $\mathrm{cost}(\mathcal{R}, \nu)$, of $(\mathcal{R}, \nu)$ is the infimum of the costs of the graphings that generate $\mathcal{R}$. The ***graph (structure)*** $\Phi[x]$ of a graphing Φ at a point $x \in X$ is the graph whose vertex set is the equivalence class $\mathcal{R}_\Phi[x]$ of x and whose edges are the pairs $(y, z) \in \mathcal{R}_\Phi[x] \times \mathcal{R}_\Phi[x]$ such that for some $i \in I$, either $\varphi_i(y) = z$ or $\varphi_i(z) = y$. For more on cost, see [Gab00] or [KM04].

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Proofs

Since the union of an increasing sequence of amenable groups is still amenable, Λ contains a non-amenable finitely generated subgroup. Let Γ be such a subgroup.

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If S is a finite generating set of Γ (maybe with repetitions), $\mathcal{G} = (\mathtt{V}, \mathtt{E})$ denotes the associated right Cayley graph (with vertex set $\mathtt{V}$): The set $\mathtt{E}$ of edges is indexed by S and Γ. Given $s \in S$ and $\gamma \in \Gamma$, the corresponding edge is oriented from the vertex γ to γs. Note that Γ acts freely on $\mathcal{G}$ by multiplication on the left. Let $\rho := \mathrm{id}$, the identity of the group Γ, chosen as base vertex for $\mathcal{G}$.

The set of the subgraphs of $\mathcal{G}$ (with the same set of vertices $\mathtt{V}$) is naturally identified with $\Omega := \{0,1\}^{\mathtt{E}}$. The connected components of $\omega \in \Omega$ are called its ***clusters***.

Consider a probability-measure-preserving essentially free (left) Γ-action on some standard Borel space (X, ν) together with a Γ-equivariant Borel map $\pi : X \to \{0,1\}^{\mathtt{E}}$.

The ***full*** equivalence relation $\mathcal{R}_\Gamma$ generated by the Γ-action X is graphed by the graphing $\Phi = (\varphi_s)_{s\in S}$, where φ_s denotes the action by s^{-1}.

We define the following equivalence subrelation on X (see [Gab05, Sect. 1]): the ***cluster equivalence subrelation*** $\mathcal{R}^{\mathrm{cl}}$, graphed by the graphing $\Phi^{\mathrm{cl}} := (\varphi_s^{\mathrm{cl}})_{s\in S}$ of partial isomorphisms, where φ_s^{cl} is the restriction $\varphi_s^{\mathrm{cl}} := \varphi_s | A_s$ of φ_s to the Borel subset A_s of $x \in X$ for which the edge e labelled s from ρ to ρs lies in $\pi(x)$, i.e., $\pi(x)(e) = 1$. Consequently, *two points $x, y \in X$ are $\mathcal{R}^{\mathrm{cl}}$-equivalent if and only if there is some $\gamma \in \Gamma$ such that $\gamma^{-1}x = y$ and the vertices $\rho, \gamma\rho$ are in the same cluster of $\pi(x)$.*

The graph structure $\Phi^{\mathrm{cl}}[x]$ given by the graphing Φ^{cl} to the $\mathcal{R}^{\mathrm{cl}}$-class of any $x \in X$ is naturally isomorphic with the cluster $\pi(x)_\rho$ of $\pi(x)$ that contains the base vertex. Denote by $U^\infty \subset X$ the Borel set of $x \in X$ whose $\mathcal{R}^{\mathrm{cl}}$-class is infinite and by $\mathcal{R}_{\Gamma|\infty}$ (resp. $\mathcal{R}^{\mathrm{cl}}_{|\infty}$) the restriction of $\mathcal{R}_\Gamma$ (resp. $\mathcal{R}^{\mathrm{cl}}$) to U^∞.

Write $\mathcal{P}(Y)$ for the power set of Y. The map $X \times \mathtt{V} \to X$ defined by $(x, \gamma\rho) \mapsto \gamma^{-1}.x$ induces a map $\Psi : X \times \mathcal{P}(\mathtt{V}) \to \mathcal{P}(X)$ that is invariant under the (left) diagonal Γ-action (i.e., $\Psi(\gamma.x, \gamma.C) = \Psi(x, C)$ for all $\gamma \in \Gamma$, $x \in X$, and $C \subset \mathtt{V}$) and such that $\Psi(x, \mathtt{V})$ is the whole $\mathcal{R}_\Gamma$-class of x. The restriction of Ψ to the Γ-invariant subset $\mathfrak{C}^{\mathrm{cl}}_\infty := \{(x, C) : x \in X, C \in \mathcal{P}(\mathtt{V}), C \text{ is an infinite cluster of } \pi(x)\}$ sends each (x, C) (and its Γ-orbit) to a whole infinite $\mathcal{R}^{\mathrm{cl}}$-class, namely, the $\mathcal{R}^{\mathrm{cl}}$-class of $\gamma^{-1}.x$ for any γ such that $\gamma\rho \in C$. Moreover, for each $x \in U^\infty$, its $\mathcal{R}_{\Gamma|\infty}$-class decomposes into infinite $\mathcal{R}^{\mathrm{cl}}$-sub-classes that are in one-to-one correspondence with the elements of $\mathfrak{C}^{\mathrm{cl}}_\infty$ that have x as first coordinate. Note that the set $\{(x, y, C) \in X \times X \times \mathcal{P}(\mathtt{V}) : x \in \Psi(y, C)\}$ is Borel, whence for a Borel set $\mathcal{A} \subset \mathfrak{C}^{\mathrm{cl}}_\infty$, the set $\overline{\Psi}(\mathcal{A}) := \bigcup \Psi(\mathcal{A})$ is measurable, being the projection onto the first coordinate of the Borel set $\{(x, y, C) : x \in \Psi(y, C)\} \cap (X \times \mathcal{A})$.

We say that (ν, π) has ***indistinguishable infinite clusters*** if for every Γ-invariant Borel subset $\mathcal{A} \subset \mathfrak{C}^{\mathrm{cl}}_\infty$, the set of $x \in X$ for which some $(x, C) \in \mathcal{A}$ and some $(x, C) \in \mathfrak{C}^{\mathrm{cl}}_\infty \setminus \mathcal{A}$ has ν-measure 0. In other words, the $\mathcal{R}^{\mathrm{cl}}$-invariant partition $U^\infty = \overline{\Psi}(\mathcal{A}) \cup \complement\overline{\Psi}(\mathcal{A})$ is not allowed to split any $\mathcal{R}_{\Gamma|\infty}$-class (up to a union of measure 0 of such classes). The following proposition, using this refined notion of indistinguishability, corrects [Gab05, Rem. 2.3].

Proposition 5 *Let Γ act ergodically on (X,ν) and $\pi : X \to \{0,1\}^{\mathtt{E}}$ be a Γ-equivariant Borel map such that $\nu(U^\infty) \neq 0$. Then $\mathcal{R}^{\mathrm{cl}}_{|\infty}$, the cluster equivalence relation restricted to its infinite locus U^∞, is ergodic if and only if (ν,π) has indistinguishable infinite clusters.*

Proof. Suppose that $\mathcal{R}^{\mathrm{cl}}_{|\infty}$ is ergodic. Then for every Γ-invariant Borel subset $\mathcal{A} \subset \mathfrak{C}^{\mathrm{cl}}_\infty$, its image $\overline{\Psi}(\mathcal{A})$ is a union of $\mathcal{R}^{\mathrm{cl}}_{|\infty}$-classes, whence $\overline{\Psi}(\mathcal{A})$ or its complement $\complement\overline{\Psi}(\mathcal{A})$ in U^∞ has measure 0. In particular, the partition $U^\infty = \overline{\Psi}(\mathcal{A}) \cup \complement\overline{\Psi}(\mathcal{A})$ is trivial, whence ν has indistinguishable infinite clusters.

Conversely, suppose that (ν,π) has indistinguishable infinite clusters. An $\mathcal{R}^{\mathrm{cl}}_{|\infty}$-invariant partition $U^\infty = \mathcal{U} \cup \complement\mathcal{U}$ defines a partition $\mathfrak{C}^{\mathrm{cl}}_\infty = \mathcal{A} \cup \complement\mathcal{A}$ according to whether $\Psi(x,C) \in \mathcal{U}$ or $\complement\mathcal{U}$. Then for ν-almost every $x \in U^\infty$, all $\Psi(x,C)$ are in $\mathcal{U}$ or all are in its complement, i.e., the $\mathcal{R}^{\mathrm{cl}}_{|\infty}$-subclasses into which the $\mathcal{R}_{\Gamma|\infty}$-class of x splits all belong to one side. Since $\mathcal{R}_{\Gamma|\infty}$ is ν-ergodic, this side has to be the same for almost every x. This means that the other side is a null set. This holds for any partition $\mathcal{U} \cup \complement\mathcal{U}$, whence $\mathcal{R}^{\mathrm{cl}}_{|\infty}$ is ergodic. ∎

If X has the form $X = \Omega \times Y$, then ν is called ***insertion tolerant*** (see [LS99]) if for each edge $e \in \mathtt{E}$, the map $\Pi_e : X \to X$ defined by $(\omega, y) \mapsto (\omega \cup \{e\}, y)$ quasi-preserves the measure, i.e., $\nu(A) > 0$ implies $\nu(\Pi_e(A)) > 0$ for every measurable subset $A \subseteq X$. Call a map $\pi : X \to \Omega$ ***increasing*** if $\pi(\omega, y) \supseteq \omega$ for all $\omega \in \Omega$. An action of Γ on $\Omega \times Y$ is always assumed to act on the first coordinate in the usual way. A slight extension of [LS99, Th. 3.3, Rem. 3.4], proved in the same way, is the following:

Proposition 6 *Assume that Γ acts on $(\Omega \times Y, \nu)$ preserving the measure and $\pi : \Omega \times Y \to \Omega$ is an increasing Γ-equivariant Borel map with $\nu(U^\infty) \neq 0$. If ν is insertion tolerant, then (ν,π) has indistinguishable infinite clusters.*

Proposition 7 *If $\Gamma < \Lambda$, then there are Γ-equivariant isomorphisms $([0,1]^{\mathtt{E}}, \mu) \simeq ([0,1]^{\Gamma}, \mu) \simeq ([0,1]^{\Lambda}, \mu)$ between the Bernoulli shift actions of Γ. In particular, the orbits of the Bernoulli shift Λ-action on $[0,1]^\Lambda$ are partitioned into subsets that are identified with the orbits of the standard Bernoulli shift Γ-action on $[0,1]^\Gamma$.*

Proof. A countable set $\mathtt{E}$ on which Γ acts freely may be decomposed by choosing a representative in each orbit so as to be identified with a disjoint union of Γ-copies, $\mathtt{E} \simeq \coprod_J \Gamma$, and to give Γ-equivariant identifications $[0,1]^{\mathtt{E}} = [0,1]^{\coprod_J \Gamma} = ([0,1]^J)^\Gamma$.

The edge set $\mathtt{E} \simeq \coprod_S \Gamma$ of the Cayley graph of Γ, as well as $\Lambda \simeq \coprod_I \Gamma$ are such countable Γ-sets. Then isomorphisms of standard Borel probability spaces $([0,1], Leb) \simeq ([0,1]^S, \otimes_S Leb) \simeq ([0,1]^I, \otimes_I Leb)$ induce Γ-equivariant isomorphisms of the Bernoulli shifts:

$$\begin{array}{ccccc} [0,1]^\Gamma & \simeq & ([0,1]^S)^\Gamma & \simeq & ([0,1]^I)^\Gamma \\ \| & & \| & & \| \\ [0,1]^\Gamma & \simeq & [0,1]^{\mathtt{E}} & \simeq & [0,1]^\Lambda . \end{array}$$

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A subgraph $(\mathtt{V}', \mathtt{E}')$ of a graph $(\mathtt{V}, \mathtt{E})$ is called ***spanning*** if $\mathtt{V}' = \mathtt{V}$. A vertex a in a graph is called a ***cutvertex*** if there are two other vertices in its component with the property that every path joining them passes through a. A ***block*** of a graph is a maximal connected subgraph that has no cutvertex. Every simple cycle of a graph is contained within one of its blocks.

Lemma 8 *If all vertices of a block have finite degree and for each pair of vertices (a, b) there are only finitely many distinct paths joining a to b, then the block is finite.*

Proof. Suppose for a contradiction that the block is infinite. Then it contains a simple infinite path P of vertices $a_1, a_2, \ldots$. By Menger's theorem, a_1 and a_n belong to a simple cycle C_n for each $n > 1$. But this implies that there are infinitely many distinct paths joining a_1 to a_2: Fix n and let a_j $(2 \le j \le n)$ be the vertex on $P \cap C_n$ with minimal index j. We may assume that C_n is oriented so that it visits a_n before it visits a_j. Then simply follow C_n from a_1 until a_j, and then follow P to a_2. ∎

Proposition 9 *(For any Cayley graph) Let $X := \Omega \times [0, 1]^\Gamma$ and $\epsilon > 0$. Let $\nu := \mu_\epsilon \times \mu$. There is a Γ-equivariant Borel map $f : X \to \Omega$ such that (ν, f) has indistinguishable infinite clusters and for all sufficiently small ϵ, the push-forward measure $f_*(\nu)$ of ν is supported on the set of spanning subgraphs of $\mathcal{G}$ each of whose components is a tree with infinitely many ends.*

Proof. We may equivariantly identify $([0, 1]^\Gamma, \mu)$ with $([0, 1]^{\mathbb{N}\times\Gamma} \times [0, 1]^{\mathtt{E}}, \mu \times \mu)$, so we identify (X, ν) with $(\Omega \times [0, 1]^{\mathbb{N}\times\Gamma} \times [0, 1]^{\mathtt{E}}, \mu_\epsilon \times \mu \times \mu)$. Fix an ordering of $S \coprod S^{-1}$; this determines an ordering of the edges incident to each vertex in $\mathcal{G}$, where we ignore edge orientations for the rest of this proof. With d denoting the degree of $\mathcal{G}$, define the function $h(t) := \lceil dt \rceil$ for $t \in [0, 1]$. Given a point $x = (\omega, (r(n, \gamma))_{n\in\mathbb{N},\gamma\in\Gamma}, u) \in X$, construct the wired spanning forest $\mathcal{F}_1$ of $\mathcal{G}$ by using the cycle-popping algorithm of D. Wilson [Wil96, Sect. 3] as adapted in [BLPS01, Th. 5.1], also called there "Wilson's algorithm rooted at infinity", applied to the stacks where the nth edge in the stack under γ is defined as the $h(r(n, \gamma))$th edge incident to γ. The measure ν is insertion tolerant and the map $\pi : x \mapsto \omega \cup \mathcal{F}_1$ is increasing, whence by Proposition 6, the pair (ν, π) has indistinguishable infinite clusters. Notice that all clusters are infinite. Now use u to construct the free minimal spanning forest $\mathcal{F}_2$ in each cluster of $\pi(x)$, that is, for every cycle $\Delta \subset \pi(x)$, delete the edge $e \in \Delta$ with maximum $u(e)$ in that cycle. The map f is $f(x) := \mathcal{F}_2$.

Now the ν-expected number of distinct simple paths in $\pi(x)$ that join any two vertices is finite (equation (13.7) of [BLPS01]) for all sufficiently small ϵ. In particular, the number of such paths is finite ν-a.s. By Lemma 8, this means that all blocks of $\pi(x)$ are finite, so that $\mathcal{F}_2$ is a spanning tree in each block. Therefore each component of $\mathcal{F}_2$ spans a component of $\pi(x)$. Thus, $\mathcal{F}_2$ determines the same cluster relation and so (ν, f) also has indistinguishable (infinite) clusters. Finally, the fact that the clusters of $\pi(x)$, and hence those of $\mathcal{F}_2$, have infinitely many ends follows, e.g., from [BLPS01, Th. 13.7]. ∎

The cluster relation determined by f of Proposition 9 is treeable and has cost larger than 1 by [Gab00, Cor. IV.24 (2)], has finite cost (since the degree is bounded), and is ergodic by Proposition 5. Since we may equivariantly identify $(\Omega \times [0,1]^\Gamma, \mu_\epsilon \times \mu)$ with $([0,1]^\Gamma, \mu)$, we proved:

Proposition 10 *For any Cayley graph of* Γ*, the Bernoulli action on* $([0,1]^\Gamma, \mu)$ *contains a treeable subrelation that is ergodic and has cost in the interval* $(1, \infty)$*.*

At this point, we already have a reasonable answer to the analogue of von Neumann's problem, since "treeable relation" is the analogue of "free group" and cost $\mathcal{C} > 1$ is, in the context of treeable relations, equivalent to non-amenability.

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An alternative approach begins with a more explicit f and a more common measure $f_*(\mu)$, namely, the Bernoulli measure μ_p on $\{0,1\}^{\mathtt{E}}$ for a certain parameter p, but requires us to choose a particular Cayley graph for Γ. It also requires us to obtain a treeable subrelation in a less explicit way. This is accomplished as follows.

Results of Häggström-Peres [HP99] imply that there are two critical values $0 < p_c \leq p_u \leq 1$ such that
- (finite phase, $p \in [0, p_c)$) μ_p-a.s., the subgraph has only finite clusters;
- (***non-uniqueness phase***, $p \in (p_c, p_u)$) μ_p-a.s., infinitely many of the clusters of the subgraph are infinite, each one with infinitely many ends;
- (the uniqueness phase, $p \in (p_u, 1]$) μ_p-a.s., the subgraph has only one cluster that is infinite.

The situation for the critical values p_c and p_u themselves is far from clear. Benjamini and Schramm [BS96] conjectured that $p_c \neq p_u$ for every Cayley graph of a f.g. non-amenable group. The main result of [PSN00] (Th. 1, p. 498) asserts that given a f.g. non-amenable group Γ, there is a finite set of generators such that the associated Cayley graph admits a non-trivial interval of non-uniqueness. Thus:

Proposition 11 *(For particular Cayley graphs) There exists a Cayley graph of* Γ *and a non-empty interval* (p_c, p_u) *such that, for any* $p \in (p_c, p_u)$*, the Bernoulli measure* μ_p *on* $\{0,1\}^{\mathtt{E}}$ *is supported on the set of subgraphs admitting infinite components, each one with infinitely many ends.*

Let $\pi : (X, \nu) \to \{0,1\}^{\mathtt{E}}$ denote either
(i) $f_p : ([0,1]^{\mathtt{E}}, \mu) \to \{0,1\}^{\mathtt{E}}$ induced by the characteristic function $\chi_{[0,p]} : [0,1] \to \{0,1\}$ of $[0,p]$, or
(ii) the identity map $(\{0,1\}^{\mathtt{E}}, \mu_p) \to \{0,1\}^{\mathtt{E}}$,
both with the natural Bernoulli Γ-action. Notice that the action is essentially free when $0 < p < 1$.

In case (ii), we have that (μ_p, π) has indistinguishable infinite clusters by [LS99, Th. 3.3]. Case (i) is essentially the same, but first we must identify $([0,1]^{\mathtt{E}}, \mu)$ equivariantly as $(\{0,1\} \times [0,1])^{\mathtt{E}} = \{0,1\}^{\mathtt{E}} \times [0,1]^{\mathtt{E}}$ equipped with the product measure $\mu_p \times \mu$ in such a way that f_p becomes the identity on the first coordinate. Then we have insertion tolerance and so, by [LS99, Rem. 3.4], indistinguishable infinite clusters.

Hence, in both cases, for any p given by Prop. 11, the locus U^{∞} of infinite classes of $\mathcal{R}^{\mathrm{cl}}$ is non-null and we have ergodicity of the restriction $\mathcal{R}^{\mathrm{cl}}_{|\infty}$ of $\mathcal{R}^{\mathrm{cl}}$ to U^{∞} by Proposition 5. We claim that its normalized cost (i.e., computed with respect to the normalized probability measure $\nu/\nu(U^{\infty})$ on U^{∞}) satisfies $1 < \mathcal{C}(\mathcal{R}^{\mathrm{cl}}_{|\infty}) < \infty$. The finiteness of the cost is clear since S, the index set for Φ^{cl}, is finite. That it is strictly greater than 1 is a direct application of [Gab00, Cor. IV.24 (2)], since the graph $\Phi^{\mathrm{cl}}[x] \simeq \pi(x)_{\rho}$ associated with almost every $x \in U^{\infty}$ has at least 3 ends.

In order to extend $\mathcal{R}^{\mathrm{cl}}_{|\infty}$ to a subrelation of $\mathcal{R}_{\Gamma}$ defined on the whole of X, choose an enumeration $\{\gamma_i\}_{i\in\mathbb{N}}$ of Γ. For each $x \in X \setminus U^{\infty}$, let γ_x be the first element $\gamma_j \in \Gamma$ such that $\gamma_j \cdot x \in U^{\infty}$. Then the smallest equivalence relation containing $\mathcal{R}^{\mathrm{cl}}_{|\infty}$ and the $(x, \gamma_x \cdot x)$'s is a subrelation of $\mathcal{R}_{\Gamma}$, is ergodic, and has cost in $(1,\infty)$ by the induction formula of [Gab00, Prop. II.6]. We proved:

Proposition 12 *For a Cayley graph and a p given by Proposition 11, the Bernoulli actions on both $([0,1]^{\mathtt{E}}, \mu)$ and $(\{0,1\}^{\mathtt{E}}, \mu_p)$ contain a subrelation that is ergodic and has cost in the open interval $(1,\infty)$.*

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Proposition 13 *If an equivalence relation $\mathcal{R}$ is ergodic and has cost in $(1,\infty)$, then it contains a treeable subrelation $\mathcal{S}_1$ that is ergodic and has cost in $(1,\infty)$.*

Proof. This is ensured by a result proved independently by A. Kechris and B. Miller [KM04, Lem. 28.11; 28.12] and by M. Pichot [Pic05, Cor. 40], through a process of erasing cycles from a graphing of $\mathcal{S}_1$ with finite cost that contains an ergodic global isomorphism. ■

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Proposition 14 *If a treeable equivalence relation $\mathcal{S}_1$ is ergodic and has cost > 1, then it contains a.e. a subrelation $\mathcal{S}_2$ that is generated by an ergodic free action of the free group $\mathbf{F}_2$.*

Proof. If the cost of $\mathcal{S}_1$ is > 2, this follows from a result of G. Hjorth [Hjo06] (see also [KM04, Sect. 28]). Otherwise, one first considers the restriction of the treeable $\mathcal{S}_1$ to a small enough Borel subset V: this increases the normalized cost by the induction formula of [Gab00, Prop. II.6 (2)] to get $\mathcal{C}(\mathcal{S}_1|V) \geq 2$. In fact, it follows from the proof of [KM04, Th. 28.3] that *a treeable probability measure-preserving equivalence relation with cost ≥ 2 contains a.e. an equivalence subrelation that is given by a free action of the free group $\mathbf{F}_2 = \langle a, b\rangle$ in such a way that the generator a acts ergodically.* By considering a subgroup of $\mathbf{F}_2$ generated by a and some conjugates of a of the form $b^k a b^{-k}$, one gets an ergodic treeable subrelation of $\mathcal{S}_1|V$ with a big enough normalized cost that, when extended to the whole of X (by using partial isomorphisms of $\mathcal{S}_1$), it gets cost ≥ 2 (by the induction back [Gab00, Prop. II.6 (2)]) and of course remains ergodic. Another application of the above-italicized result gives the desired ergodic action of $\mathbf{F}_2$ on X. ■

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The proof of Theorem 2 is now complete as a direct consequence of Propositions 12 (for the case $X = \{0,1\}^{\mathrm{E}}$), 13 and 14. ■

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In case $X = [0,1]^{\mathrm{E}}$ of Prop. 12, by using Prop. 7, we can see $\mathcal{S}_2$ (with $\mathcal{S}_2 \subset \mathcal{S}_1 \subset \mathcal{R}_\Gamma$ given by Prop. 14 and 13) as an equivalence subrelation of that given by the Bernoulli shift action of Λ. This finishes the proof of Theorem 1. Alternatively, we may use Prop. 10 and a similar argument to prove Theorem 1. ■

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Proof of Cor 3. For any diffuse abelian subalgebra A of $L(H)$, the von Neumann factor $L(H \wr \Lambda) = L(\Lambda \ltimes \oplus_\Lambda H)$ contains the von Neumann algebra crossed product $\Lambda \ltimes \otimes_\Lambda A$, which is isomorphic with the group-measure-space factor $\Lambda \ltimes L^\infty([0,1]^\Lambda, \mu)$ associated with the Bernoulli shift. The corollary then follows from Th. 1. ■

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Proof of Cor. 4. If $\hat{K}$ is the dual group of K, then $L(H \wr \Gamma)$ contains $L(K \wr \Gamma)$, which is isomorphic with the group-measure-space factor $\Gamma \ltimes L^\infty(\hat{K}^\Gamma)$ associated with the Bernoulli shift of Γ on $\hat{K}^\Gamma$, where the finite set $\hat{K} \simeq \{1, 2, \ldots, k\}$ is equipped with the equiprobability measure ν. The result is then obtained by taking the pull-back of the $\mathbf{F}_2$-action on $\prod_1^n \{0,1\}^\Gamma$, given in Th. 2, by the Γ-equivariant Borel map $\hat{K}^\Gamma \to (\{0,1\}^n)^\Gamma \simeq \prod_1^n \{0,1\}^\Gamma$, sending $\otimes\nu$ to μ_p, that extends a map $\{1, 2, \ldots, k\} \to \{0,1\}^n$ (whose existence is ensured by the form of $k = \beta^n$). ■

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It is likely that the free minimal spanning forest (FMSF) of a Cayley graph of Γ would serve as the desired ergodic subrelation $\mathcal{S}_1$ of Prop. 13, but its indistinguishability, conjectured in [LPS06], is not known. Also, it is not known to have cost > 1, but this is equivalent to $p_c < p_u$, which is conjectured to hold and which we know holds for some Cayley graph. See [LPS06] for information on the FMSF and [Tim06] for a weak form of indistinguishability.

A general question remains open:

Question: Does every probability-measure-preserving free ergodic action of a non-amenable countable group contain an ergodic subrelation generated by a free action of a non-cyclic free group? More generally: Does every standard countable probability-measure-preserving non-amenable ergodic equivalence relation contain a treeable non-amenable ergodic equivalence subrelation?

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